

# CFT, The Yellow Book

## A Solution Manual

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### Chapter 2: Quantum Field Theory

#### Exercise 2.1

Since the operator  $\mathcal{D}$  can involve higher order derivatives, the eigenfuncions usually need degeneracy indices as well.

$$\int dx u_{n,s}^* u_{m,r} = \delta_{mn} \delta_{rs}$$

First define

$$\tilde{\phi}_{n,s} \equiv \int dx u_{n,s}^*(x) \phi(x)$$

$$\tilde{\pi}_{n,s} \equiv \int dx u_{n,s}^*(x) \pi(x)$$

These operators are such that  $\tilde{\phi}_{n,s}$  and  $\tilde{\pi}_{n,s}^\dagger$  are conjugate. Furthermore the Hamiltonian becomes

$$\begin{aligned} H &= \frac{1}{2} \int dx (\pi^2 - \phi \mathcal{D} \phi) \\ &= \frac{1}{2} \sum_{n,s} (\tilde{\pi}_{n,s}^\dagger \tilde{\pi}_{n,s} + \omega_n^2 \tilde{\phi}_{n,s}^\dagger \tilde{\phi}_{n,s}). \end{aligned}$$

Finally define

$$a_{n,s} \equiv \frac{\omega_n \tilde{\phi}_{n,s} + i \tilde{\pi}_{n,s}}{\sqrt{2\omega_n}}$$

these operators satisfy

$$\boxed{[a_{n,s}, a_{m,r}^\dagger] = \delta_{mn} \delta_{rs}}$$

Also, note that if  $u_{n,s}$  form a complete basis, then so do  $u_{n,s}^*$ ; this implies

$$\boxed{\phi(x) = \sum_{n,s} \frac{1}{\sqrt{2\omega_n}} (u_{n,s}(x) a_{n,s} + u_{n,s}^*(x) a_{n,s}^\dagger)}$$

finally, we write

$$\omega_n a_{n,s}^\dagger a_{n,s} = H_{n,s} - \frac{i}{2} \omega_n (\tilde{\pi}_{n,s}^\dagger \tilde{\phi}_{n,s} - \tilde{\phi}_{n,s}^\dagger \tilde{\pi}_{n,s}) = H_{n,s} - \frac{1}{2} \omega_n$$

therefore  $H = \sum_{n,s} H_{n,s}$  becomes

$$H = \sum_{n,s} \omega_n \left( a_{n,s}^\dagger a_{n,s} + \frac{1}{2} \right)$$

## Exercise 2.2

Under a change of variables  $\phi(x) \rightarrow \phi'(x) = \phi(x) + \varepsilon \delta \phi(x)$ , the value of an integral remains unchanged

$$\begin{aligned} \langle X \rangle &= \frac{1}{Z} \int [d\phi] X e^{iS[\phi]} = \frac{1}{Z} \int [d\phi'] X' e^{iS'[\phi']} \\ &= \frac{1}{Z} \int [d\phi] \left| \frac{\partial \phi}{\partial \phi'} \right| (X + \delta X) e^{iS + i\delta S} \end{aligned}$$

Whenever  $\delta \phi(x)$  does *not* depend on the field  $\phi(x)$ , the Jacobian becomes unity and, up to the first order (in  $\varepsilon$ ), we get

$$\langle \delta X \rangle + i \langle X \delta S \rangle = 0$$

For  $X = \phi(y)$ , and  $\delta \phi(x) = \delta(x - z)$ , we have  $\delta X = \varepsilon \delta(y - z)$  and

$$\delta S = - \int dx \left( m^2 \phi(x) \delta(x - z) + \phi'^\mu(x) \partial_\mu \delta(x - z) \right) = (\square - m^2) \phi(z)$$

which yields<sup>1</sup>

$$(\square_z - m^2) \langle \phi(y) \phi(z) \rangle = i \delta(y - z)$$

## Exercise 2.3

REMARK: This is only valid for even potentials.

Essentially, we need to show

$$\{\psi_i, \psi_j\} \frac{\partial V}{\partial \psi_j} \stackrel{!}{=} [\psi_i, V(\psi)]$$

For the trivial even term, i.e. a constant potential, this obviously holds. For a generic even term and using the fact that the anti-commutator is a c-number, we may write

$$\begin{aligned} & \{\psi_i, \psi_j\} \partial_j \psi_{k_1} \psi_{l_1} \cdots \psi_{k_n} \psi_{l_n} \\ &= \sum_{r=1}^n \delta_{jk_r} \psi_{k_1} \psi_{l_1} \cdots \psi_{k_{r-1}} \psi_{l_{r-1}} \{\psi_i, \psi_j\} \psi_{l_r} \psi_{k_{r+1}} \psi_{l_{r+1}} \cdots \psi_{k_n} \psi_{l_n} \\ & \quad - \sum_{r=1}^n \delta_{jl_r} \psi_{k_1} \psi_{l_1} \cdots \psi_{k_{r-1}} \psi_{l_{r-1}} \psi_{k_r} \{\psi_i, \psi_j\} \psi_{k_{r+1}} \psi_{l_{r+1}} \cdots \psi_{k_n} \psi_{l_n} \\ &= \sum_{r=1}^n \psi_{k_1} \psi_{l_1} \cdots \psi_{k_{r-1}} \psi_{l_{r-1}} (\psi_i \psi_{k_r} \psi_{l_r} + \psi_{k_r} \psi_i \psi_{l_r} - \psi_{k_r} \psi_i \psi_{l_r} - \psi_{k_r} \psi_{l_r} \psi_i) \psi_{k_{r+1}} \psi_{l_{r+1}} \cdots \psi_{k_n} \psi_{l_n} \\ &= [\psi_i, \psi_{k_1} \psi_{l_1} \cdots \psi_{k_n} \psi_{l_n}] \blacksquare \end{aligned}$$

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<sup>1</sup>Note that I am using a  $-++$  signature.

## Exercise 2.4

To be added!

## Exercise 2.5

The relevant term is

$$\begin{aligned}\langle \theta_{i_1} \theta_{i_2} \theta_{i_3} \theta_{i_4} \rangle &= \frac{1}{8} A_{j_1 j_2}^{-1} A_{j_3 j_4}^{-1} \partial_{i_4} \partial_{i_3} \partial_{i_2} \partial_{i_1} b_{j_1} b_{j_2} b_{j_3} b_{j_4} \\ &= \frac{1}{8} \langle \theta_{j_1} \theta_{j_2} \rangle \langle \theta_{j_3} \theta_{j_4} \rangle \sum_{p \in S_4} (-)^{\sigma(p)} \delta_{j_1 i_{p(1)}} \delta_{j_2 i_{p(2)}} \delta_{j_3 i_{p(3)}} \delta_{j_4 i_{p(4)}}\end{aligned}$$

where  $\sigma(p)$  is the number of adjacent permutations in the permutation  $p$ . Each Wick grouping gets repeated 8 times, one factor of two for internal order of each grouping and an extra factor of two for the order of the pairs. Symmetry of the ordinary multiplication and the antisymmetry of Grassmann multiplication along with antisymmetry of  $A_{ij}^{-1}$ , makes all these terms have the same sign and therefore

$$\langle \theta_{i_1} \theta_{i_2} \theta_{i_3} \theta_{i_4} \rangle = \pm \langle \theta_{i_1} \theta_{i_2} \rangle \langle \theta_{i_3} \theta_{i_4} \rangle \pm \langle \theta_{i_1} \theta_{i_3} \rangle \langle \theta_{i_2} \theta_{i_4} \rangle \pm \langle \theta_{i_1} \theta_{i_4} \rangle \langle \theta_{i_2} \theta_{i_3} \rangle$$

counting the adjacent permutations, the signs turn out to be

$$\boxed{\langle \theta_{i_1} \theta_{i_2} \theta_{i_3} \theta_{i_4} \rangle = + \langle \theta_{i_1} \theta_{i_2} \rangle \langle \theta_{i_3} \theta_{i_4} \rangle - \langle \theta_{i_1} \theta_{i_3} \rangle \langle \theta_{i_2} \theta_{i_4} \rangle + \langle \theta_{i_1} \theta_{i_4} \rangle \langle \theta_{i_2} \theta_{i_3} \rangle}$$

## Exercise 2.6

Before starting to solve the problem, let us mention that the order of appearance of variables in  $d\bar{\theta}$  is the opposite of the order in  $d\theta$ ; so

$$\int d\bar{\theta} d\theta \equiv \frac{\partial}{\partial \bar{\theta}_1} \cdots \frac{\partial}{\partial \bar{\theta}_n} \frac{\partial}{\partial \theta_n} \cdots \frac{\partial}{\partial \theta_1}.$$

Now, back to the problem at hand, let us start by writing

$$e^{-M_{ij} \bar{\theta}_i \theta_j} = \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \sum_{\substack{i_1 \cdots i_n \\ j_1 \cdots j_n}} (M_{i_1 j_1} \bar{\theta}_{i_1} \theta_{j_1}) \cdots (M_{i_n j_n} \bar{\theta}_{i_n} \theta_{j_n})$$

One can permute the parantheses without changing the sign and this will cancel the  $n!$  in the denominator.

$$e^{-M_{ij} \bar{\theta}_i \theta_j} = \sum_{n=0}^{\infty} (-)^n \sum_{\langle (i_1, j_1), \dots, (i_n, j_n) \rangle} (M_{i_1 j_1} \bar{\theta}_{i_1} \theta_{j_1}) \cdots (M_{i_n j_n} \bar{\theta}_{i_n} \theta_{j_n})$$

where the notation used in the second sum means that the sum is over unordered tuples of ordered pairs. Considering that no second powers exist in the realm of Grassmann variables, this is clearly the same as

$$e^{-M_{ij} \bar{\theta}_i \theta_j} = \prod_{(i,j)} (1 - M_{ij} \bar{\theta}_i \theta_j)$$

After taking the full integral (derivative) only the terms in which every variable is present exactly once, survives. That is

$$\int d\bar{\theta} d\theta e^{-M_{ij} \bar{\theta}_i \theta_j} = \sum_{p \in S_n} \pm M_{p(1)1} \cdots M_{p(n)n}$$

To compute the signs, first (freely) order each term by their non-barred index to get

$$(-)^n M_{p(1)1} \cdots M_{p(n)n} \int d\bar{\theta} d\theta \bar{\theta}_{p(1)} \theta_1 \cdots \bar{\theta}_{p(n)} \theta_n$$

$$= (-)^{\sigma(p)} M_{1p(1)} \cdots M_{np(n)}$$

Here is how the *minus-sign-counting* works: first, we have  $n$  minus signs from the product of all the  $(1 - M_{ij}\bar{\theta}_i\theta_j)$  terms; then we have  $1 + 2 + \cdots + n$  minus signs used to bring each  $\theta_i$  to the far left before integration; and then at last, there are  $\sigma(p) + 0 + 1 + \cdots + (n - 1)$  minus signs to order the barred variables properly. Finally we recognize this as

$$I_2(M) = \det M$$

## Chapter 3: Statistical Mechanics

### Exercise 3.1

a)

$$\begin{aligned}\langle N_L \rangle &= \sum_n \frac{N!}{n!(N-n)!} 2^{-N} \cdot n = 2^{-N} N \sum_{m:=n-1} \frac{(N-1)!}{m!(N-1-m)!} \\ &= 2^{-N} N \cdot 2^{N-1} = \boxed{N/2}\end{aligned}$$

b)

$$\begin{aligned}\langle N_L^2 \rangle &= \langle N_L \rangle + \langle N_L(N_L - 1) \rangle \\ &= \frac{N}{2} + \sum_n \frac{N!}{n!(N-n)!} 2^{-N} \cdot n(n-1) \\ &= \frac{N}{2} + 2^{-N} N(N-1) \sum_{m=n-2} \frac{(N-2)!}{m!(N-2-m)!} \\ &= \frac{N}{2} + \frac{N(N-1)}{4} = \frac{N(N+1)}{4}\end{aligned}$$

therefore

$$\boxed{\Delta N_L = \sqrt{\langle N_L^2 \rangle - \langle N_L \rangle^2} = \frac{\sqrt{N}}{2}}$$

c) Defining

$$X \equiv \frac{N_L - N/2}{\sqrt{N}/2}$$

and in the large  $N$  limit, we may write

$$f_X(x) \frac{2}{\sqrt{N}} \approx \mathbb{P}[N_L = n(x)]$$

where

$$n(x) \equiv \left\lfloor \frac{N + x\sqrt{N}}{2} \right\rfloor \approx \frac{N + x\sqrt{N}}{2}$$

Using the Stirling formula (note that  $\lim_{N \rightarrow \infty} n(x) = \infty$ ) we get

$$\boxed{f_X(x) = \frac{1}{\sqrt{2\pi}e^{x^2/2}}}$$

### Exercise 3.2

a)

$$\begin{aligned}Z &= \sum_{[\sigma]} \exp \left\{ \sum_{n=0}^{N-1} [-(H/2)(\sigma_n + \sigma_{n+1}) + K\sigma_n\sigma_{n+1}] \right\} \\ &= \sum_{[\sigma]} T_{\sigma_n, \sigma_{n+1}} = \text{Tr } T^N\end{aligned}$$

with

$$T_{\sigma, \sigma'} \equiv \exp \left\{ \sum_{n=0}^{N-1} [-(H/2)(\sigma + \sigma') + K\sigma\sigma'] \right\}$$

or, in matrix form

$$T \equiv \begin{pmatrix} e^{K-H} & e^{-K} \\ e^{-K} & e^{K+H} \end{pmatrix}$$

b)

$$\begin{aligned} f &\equiv \lim_{N \rightarrow \infty} -\frac{1}{N} \log \text{Tr } T^N \\ &= -\log \left[ \lim_{N \rightarrow \infty} (\tau_+^N + \tau_-^N)^{1/N} \right] \\ &= -\log \tau_+ \end{aligned}$$

where

$$\tau_+ \equiv e^K \cosh H + \sqrt{e^{2K} \sinh^2 H + e^{-2K}}$$

therefore

$$f = -\log \left[ e^K \cosh H + \sqrt{e^{2K} \sinh^2 H + e^{-2K}} \right]$$

c)

$$M \equiv \frac{\partial f}{\partial H} = -\frac{e^K \sinh H + e^{2K} \sinh(2H)/2\sqrt{e^{2K} \sinh^2 H + e^{-2K}}}{e^K \cosh H + \sqrt{e^{2K} \sinh^2 H + e^{-2K}}}$$

for  $H \ll 1$

$$M \approx H e^{2K}$$

and therefore the magnetic susceptibility diverges like  $e^{2K}$  as temperature approaches zero. We also note that at finite temperatures, the free energy (per spin site) is well behaved and therefore we observe no phase transitions.

d) Using the symmetries of the problem, it is easy to show that the two point correlation function

$$C_\ell \equiv \langle \sigma_n \sigma_{n+\ell} \rangle - \langle \sigma^2 \rangle$$

is, up to a constant function of  $\ell$ , the same as

$$2 \left[ P_{++}^\ell(H, K) + P_{++}^\ell(-H, K) \right]$$

where  $P_{++}^\ell(H, K)$  is the probability that two spins, a distance  $\ell$  apart, are both pointing up. This is written as

$$\begin{aligned} P_{++}^\ell(H, K) &= \lim_{N \rightarrow \infty} \frac{\langle + | T^\ell | + \rangle \langle + | T^N | + \rangle}{\text{Tr } T^{N+\ell}} \\ &= \left| \langle \tau_+ | + \rangle \langle + | \tau_- \rangle \right|^2 \left( \frac{\tau_-}{\tau_+} \right)^\ell + cte. \end{aligned}$$

I won't substitute the complete formulae; only note that since the spectrum of  $T$  does not depend on the sign of  $H$ , we know that the correlation function is always in the form

$$C_\ell = C_0 \left( \frac{1 - \chi}{1 + \chi} \right)^\ell$$

where

$$\chi \equiv \sqrt{\tanh^2 H + e^{-4K} (1 - \tanh^2 H)}$$